HOMEWORK 5

Problem 1 Prove that for $\operatorname{Re} s > 1$,

$$\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}} dx,$$

where $[x]$ stands for the integer part of x.

Problem 2 Prove that for $\operatorname{Re} s > 1$,

$$\frac{1}{\zeta(s)} = s \int_1^\infty \frac{M(x)}{x^{s+1}} dx,$$

where

$$M(x) = \sum_{1 \le n \le x} \mu(n),$$

and $\mu(n)$ is the Möbius function.

Problem 3 Prove that for $\operatorname{Re} s > 1$,

$$L(s, \chi) = s \int_1^\infty \frac{A(x)}{x^{s+1}} dx,$$

where

$$A(x) = \sum_{1 \le n \le x} \chi(n).$$

Problem 4 Suppose that the series $\sum_{n=1}^\infty a_n$ converges with sum A, and let $A(x) = \sum_{1 \le n \le x} a_n$. Prove that the Dirichlet series

$$F(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}$$

converges for $\operatorname{Re} s > 0$ and that

$$F(s) = A - s \int_1^\infty \frac{R(x)}{x^{s+1}} dx,$$

where $R(x) = A - A(x)$. (*Hint to Problems 1-4: Use Abel's summation*).

Problem 5 Function $f : \mathbb{N} \to \mathbb{C}$ is called *completely multiplicative*, if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$. Suppose that the Dirichlet series $F(s) = \sum_{n=1}^\infty \frac{f(n)}{n^s}$ converges absolutely for $\operatorname{Re} s > \sigma_0$. Prove that for $\operatorname{Re} s > \sigma_0$ we have

$$\frac{F'(s)}{F(s)} = \sum_{n=1}^\infty \frac{f(n)\Lambda(n)}{n^s},$$

where prime stands for the derivative, and $\Lambda(n)$ is von Mangoldt function.